

# Critical $\mathrm{Sp}(N)$ models in $6 - \epsilon$ dimensions and higher spin dS/CFT

Lin Fei,<sup>a</sup> Simone Giombi,<sup>a</sup> Igor R. Klebanov<sup>a,b</sup> and Grigory Tarnopolsky<sup>a</sup>

<sup>a</sup>*Department of Physics, Princeton University,  
Princeton, NJ 08544, U.S.A.*

<sup>b</sup>*Princeton Center for Theoretical Science, Princeton University,  
Princeton, NJ 08544, U.S.A.*

*E-mail:* [linf@princeton.edu](mailto:linf@princeton.edu), [sgiombi@princeton.edu](mailto:sgiombi@princeton.edu),  
[klebanov@princeton.edu](mailto:klebanov@princeton.edu), [grigory@princeton.edu](mailto:grigory@princeton.edu)

**ABSTRACT:** Theories of anti-commuting scalar fields are non-unitary, but they are of interest both in statistical mechanics and in studies of the higher spin de Sitter/Conformal Field Theory correspondence. We consider an  $\mathrm{Sp}(N)$  invariant theory of  $N$  anti-commuting scalars and one commuting scalar, which has cubic interactions and is renormalizable in 6 dimensions. For any even  $N$  we find an IR stable fixed point in  $6 - \epsilon$  dimensions at imaginary values of coupling constants. Using calculations up to three loop order, we develop  $\epsilon$  expansions for several operator dimensions and for the sphere free energy  $F$ . The conjectured  $F$ -theorem is obeyed in spite of the non-unitarity of the theory. The  $1/N$  expansion in the  $\mathrm{Sp}(N)$  theory is related to that in the corresponding  $\mathrm{O}(N)$  symmetric theory by the change of sign of  $N$ . Our results point to the existence of interacting non-unitary 5-dimensional CFTs with  $\mathrm{Sp}(N)$  symmetry, where operator dimensions are real. We conjecture that these CFTs are dual to the minimal higher spin theory in 6-dimensional de Sitter space with Neumann future boundary conditions on the scalar field. For  $N = 2$  we show that the IR fixed point possesses an enhanced global symmetry given by the supergroup  $\mathrm{OSp}(1|2)$ . This suggests the existence of  $\mathrm{OSp}(1|2)$  symmetric CFTs in dimensions smaller than 6. We show that the  $6 - \epsilon$  expansions of the scaling dimensions and sphere free energy in our  $\mathrm{OSp}(1|2)$  model are the same as in the  $q \rightarrow 0$  limit of the  $q$ -state Potts model.

**KEYWORDS:** Field Theories in Higher Dimensions, Renormalization Group

**ARXIV EPRINT:** [1502.07271](https://arxiv.org/abs/1502.07271)

---

## Contents

<b>1</b>	<b>Introduction and summary</b>	<b>1</b>
<b>2</b>	<b>The IR fixed points of the cubic <math>\text{Sp}(N)</math> theory</b>	<b>4</b>
2.1	Estimating operator dimensions with Padé approximants	7
2.2	Sphere free energies	8
<b>3</b>	<b>Symmetry enhancement for <math>N = 2</math></b>	<b>9</b>

---

## 1 Introduction and summary

Among the classic models of quantum field theory, a prominent role is played by the  $O(N)$  invariant theories of  $N$  massless scalar fields  $\phi^i$ , which interact via the potential  $\frac{\lambda}{4}(\phi^i\phi^i)^2$ . For any positive  $N$  these models possess interacting IR fixed points in dimensions  $2 < d < 4$  [1]. These theories contain  $O(N)$  singlet current operators with all even spin, and when  $N$  is large the current anomalous dimensions are  $\sim 1/N$  [2]. Since all the higher spin currents are nearly conserved, the  $O(N)$  models possess a weakly broken higher spin symmetry. In the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [3–5], each spin  $s$  conserved current in a  $d$  dimensional CFT is mapped to a massless spin  $s$  gauged field in  $d + 1$  dimensional AdS space. For these reasons, it was conjectured [6] that the singlet sector of the critical  $O(N)$  models in  $d = 3$  is dual to the interacting higher spin theory in  $d = 4$  containing massless gauge fields of all even positive spin [7–10]. This minimal Vasiliev theory also contains a scalar field with  $m^2 = -2/\ell_{\text{AdS}}^2$ , and the two admissible boundary conditions on this field [11, 12] distinguish the interacting  $O(N)$  model from the free one (in the latter all the higher spin currents are conserved exactly). A review of the higher spin AdS/CFT dualities may be found in [13].

A remarkable feature of the Vasiliev theories [7–10] is that they are consistent not only in Anti-de Sitter, but also in de Sitter space. On general grounds one expects a CFT dual to quantum gravity in  $dS_4$  to be a non-unitary theory defined in three dimensional Euclidean space [14]. In [15] it was proposed that the CFT dual to the minimal higher spin theory in  $dS_4$  is the theory of an even number  $N$  of anti-commuting scalar fields  $\chi^i$  with the action

$$S = \int d^3x \left( \frac{1}{2} \Omega_{ij} \partial_\mu \chi^i \partial^\mu \chi^j + \frac{\lambda}{4} (\Omega_{ij} \chi^i \chi^j)^2 \right). \quad (1.1)$$

This theory possesses  $\text{Sp}(N)$  symmetry, and  $\Omega_{ij}$  is the invariant symplectic matrix. This model was originally introduced and studied in [16, 17], where it was shown to possess an IR fixed point in  $4 - \epsilon$  dimensions. The beta function of this model is related to that of the  $O(N)$  model via the replacement  $N \rightarrow -N$ . According to the proposal of [15], the free UV fixed point of (1.1) is dual to the minimal higher spin theory in  $dS_4$  with the Neumann

future boundary conditions on the  $m^2 = 2/\ell_{dS}^2$  bulk scalar field, and its interacting IR fixed point to the same higher spin theory but with the Dirichlet boundary conditions on the scalar field. In the latter case, the higher spin symmetry is slightly broken at large  $N$ , and the de Sitter higher spin gauge fields are expected to acquire small masses through quantum effects.<sup>1</sup> A discussion of the de Sitter boundary conditions from the point of view of the wave function of the Universe was given in [21, 22].

In this paper we consider an extension of the proposed higher spin dS/CFT correspondence [15] to higher dimensional de Sitter spaces, and in particular to  $dS_6$ . Our construction mirrors our recent work [23, 24] on the higher dimensional extensions of the higher spin AdS/CFT. It was observed long ago [25–27] that in  $d > 4$  the quartic  $O(N)$  models possess UV fixed points which can be studied in the large  $N$  expansion. The UV completion of the  $O(N)$  scalar theory in  $4 < d < 6$  was proposed in [23]; it is the cubic  $O(N)$  symmetric theory of  $N + 1$  scalar fields  $\sigma$  and  $\phi^i$ :

$$S = \int d^d x \left( \frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} g_1 \sigma (\phi^i)^2 + \frac{1}{6} g_2 \sigma^3 \right). \quad (1.2)$$

For sufficiently large  $N$ , this theory has an IR stable fixed point in  $6 - \epsilon$  dimensions with real values of  $g_1$  and  $g_2$ . The beta functions and anomalous dimensions were calculated to three loop order [23, 24],<sup>2</sup> and the available results agree nicely with the  $1/N$  expansion of the quartic  $O(N)$  model at its UV fixed point [29–33] when the quartic  $O(N)$  model is continued to  $6 - \epsilon$  dimensions. The conformal bootstrap approach to the higher dimensional  $O(N)$  model was explored in [34–36].

To extend the idea of [15] to  $dS_{d+1}$  with  $d > 4$ , we may consider non-unitary CFTs (1.1) which in  $d > 4$  possess UV fixed points for large  $N$ . The  $1/N$  expansion of operator scaling dimensions may be developed using the generalized Hubbard-Stratonovich transformation, and one finds that it is related to the  $1/N$  expansion in the  $O(N)$  models via the replacement  $N \rightarrow -N$ . In  $d = 5$  this interacting fixed point should be dual to the higher spin theory in  $dS_6$  with Neumann boundary conditions on the  $m^2 = 6/\ell_{dS}^2$  scalar field (corresponding to the conformal dimension  $\Delta = 2 + \mathcal{O}(1/N)$  on the CFT side).<sup>3</sup> In search of the UV completion of these quartic CFTs in  $4 < d < 6$ , we introduce the cubic theory of one commuting real scalar field  $\sigma$  and  $N$  anti-commuting scalar fields  $\chi^i$ :

$$S = \int d^d x \left( \frac{1}{2} \Omega_{ij} \partial_\mu \chi^i \partial^\mu \chi^j + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} g_1 \Omega_{ij} \chi^i \chi^j \sigma + \frac{1}{6} g_2 \sigma^3 \right). \quad (1.3)$$

---

<sup>1</sup>A potential difficulty with this picture is that unitarity in  $dS_{d+1}$  space requires that a massive field of spin  $s > 1$  should satisfy  $m^2 > (s-1)(s+d-3)/\ell_{dS}^2$  [18–20]. In other words, there is a finite gap between massive fields and massless ones (the latter are dual to exactly conserved currents in the CFT). However, since the masses are generated by quantum effects and are parametrically small at large  $N$ , this is perhaps not fatal for bulk unitarity. It would be interesting to clarify this further.

<sup>2</sup>The one loop beta functions of the model (1.2) in  $d = 6$  were first calculated in [28].

<sup>3</sup>The  $d = 5$  free  $\text{Sp}(N)$  model corresponds to Dirichlet scalar boundary conditions in the  $dS_6$  dual. It should be possible to extend this free  $\text{Sp}(N)$  model/ $dS$  higher spin duality to general dimensions, since the Vasiliev equations in  $(A)dS_{d+1}$  are known for all  $d$  [10].

Alternatively, we may combine the fields  $\chi^i$  into  $N/2$  complex anti-commuting scalars  $\theta^\alpha$ ,  $\alpha = 1, \dots, N/2$  [16, 17]; then the action assumes the form

$$S = \int d^d x \left( \partial_\mu \theta^\alpha \partial^\mu \bar{\theta}^\alpha + \frac{1}{2} (\partial_\mu \sigma)^2 + g_1 \sigma \theta^\alpha \bar{\theta}^\alpha + \frac{1}{6} g_2 \sigma^3 \right). \quad (1.4)$$

We study the beta functions for this theory in  $d = 6 - \epsilon$  and show that they are related to the beta functions of the theory (1.2) via the replacement  $N \rightarrow -N$ . For all  $N$  there exists an IR fixed point of the theory (1.3) with imaginary values of  $g_1$  and  $g_2$ . This is similar to the IR fixed point of the single scalar cubic field theory (corresponding to the  $N = 0$  case of our models), which was used by Fisher [37] as an approach to the Lee-Yang edge singularity. The fact that the couplings are purely imaginary makes the integrand of the path integral oscillate rapidly at large  $\sigma$ ; this should be contrasted with real couplings giving a potential unbounded from below.

Our results allow us to study the  $6 - \epsilon$  expansion of the theory (1.3) with arbitrary  $N$ , and we observe that at finite  $N$  there are qualitative differences between the  $\text{Sp}(N)$  and  $O(N)$  models which are not seen in the  $1/N$  expansion. In fact, for the  $\text{Sp}(N)$  model there is no analogue of the lower bound  $N_{\text{crit}}$  that was found in the  $O(N)$  case [23]. For the lowest value,  $N = 2$ , we observe some special phenomena. In this theory, which contains two real anti-commuting scalars, it is impossible to formulate the quartic interaction (1.1); thus, the cubic lagrangian (3.1) seems to be the only possible description of the interacting theory with global  $\text{Sp}(2)$  symmetry. Furthermore, it becomes enhanced to the supergroup  $\text{OSp}(1|2)$  because at the IR fixed point the two coupling constants are related via  $g_2^* = 2g_1^*$ . The enhanced symmetry implies that the scaling dimensions of  $\sigma$  and  $\chi^i$  are equal, and we check this to order  $\epsilon^3$ . An example of theory with  $\text{OSp}(1|2)$  symmetry is provided by the  $q \rightarrow 0$  limit of the  $q$ -state Potts model [38].<sup>4</sup> We show that the  $6 - \epsilon$  expansions of the scaling dimensions in our  $\text{OSp}(1|2)$  symmetric theory are the same as in the  $q \rightarrow 0$  limit of the  $q$ -state Potts model.<sup>5</sup> This provides strong evidence that the  $\text{OSp}(1|2)$  symmetric IR fixed point of the cubic theory (3.1) describes the second order transitions in the ferromagnetic  $q = 0$  Potts model, which exist in  $2 < d < 6$  [39].

Using the results of [40], we also compute perturbatively the sphere free energies of the models (1.3). The sphere free energy for the  $\text{OSp}(1|2)$  symmetric model is found to be the same as for the  $q = 0$  Potts model. In terms of the quantity  $\tilde{F} = \sin(\frac{\pi d}{2}) \log Z_{S^d}$ , which was introduced in [40] as a natural way to generalize the  $F$ -theorem [41–43] to continuous dimensions, we find that the RG flow in the cubic  $\text{Sp}(N)$  models in  $d = 6 - \epsilon$  satisfies  $\tilde{F}_{\text{UV}} > \tilde{F}_{\text{IR}}$  for all  $N \geq 2$ . We show that the same result holds in the model (1.1) in  $d = 4 - \epsilon$ .<sup>6</sup> This is somewhat surprising, since for non-unitary CFTs the inequality  $\tilde{F}_{\text{UV}} > \tilde{F}_{\text{IR}}$  is not always satisfied. It would be interesting to understand if this is related

<sup>4</sup>We are grateful to Giorgio Parisi for informing us about this and for important discussions.

<sup>5</sup>We thank Sergio Caracciolo for suggesting this comparison to us and for informing us about the paper [39].

<sup>6</sup>The models (1.1) and (1.3) also satisfy the  $F$ -theorem to leading order in the large  $N$  expansion (for all  $d$ ), since the leading order correction to  $\tilde{F}_{\text{UV}} - \tilde{F}_{\text{IR}}$  is of order  $N^0$ , and was shown to satisfy the  $F$ -theorem in the corresponding unitary  $O(N)$  models [40, 43, 44].

to the “pseudo-unitary” structure discussed in [17], and to the fact that these models are presumably dual to unitary higher spin gravity theories in de Sitter space.

## 2 The IR fixed points of the cubic $\text{Sp}(N)$ theory

The beta functions and anomalous dimensions for the  $\text{Sp}(N)$  symmetric model (1.3) can be obtained by replacing  $N \rightarrow -N$  in the corresponding results for the cubic  $O(N)$  model (1.2), which were computed in [23, 24] to three loop order. Indeed, writing the action in the complex basis (1.4), we see that the Feynman rules and propagators are identical to those of the  $O(N)$  theory written in the  $U(N/2)$  basis, the only difference being that the  $N/2$  complex scalars are anticommuting. Hence, for each closed loop of the  $\theta^\alpha$  we get an extra minus sign, thus explaining the replacement  $N \rightarrow -N$ .

Using the results in [23, 24], the beta functions for the  $\text{Sp}(N)$  model are then found to be:

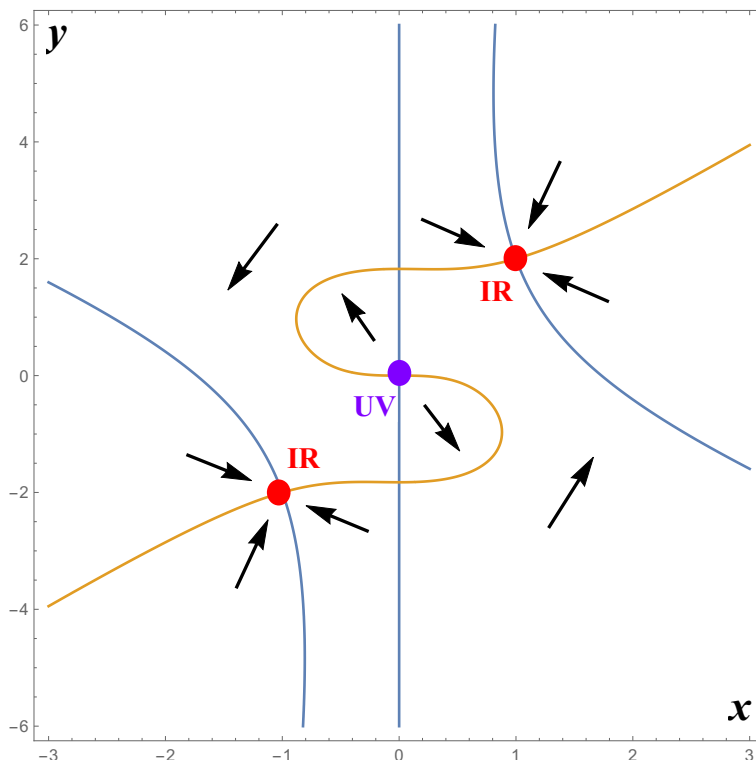
$$\begin{aligned}\beta_1 &= -\frac{\epsilon}{2}g_1 - \frac{1}{12(4\pi)^3}g_1((N+8)g_1^2 + 12g_1g_2 - g_2^2) \\ &\quad - \frac{1}{432(4\pi)^6}g_1((536-86N)g_1^4 + 12(30+11N)g_1^3g_2 + (628-11N)g_1^2g_2^2 + 24g_1g_2^3 - 13g_2^4) \\ &\quad + \dots, \\ \beta_2 &= -\frac{\epsilon}{2}g_2 + \frac{1}{4(4\pi)^3}(4Ng_1^3 - Ng_1^2g_2 - 3g_2^3) \\ &\quad + \frac{1}{144(4\pi)^6}(24Ng_1^5 + 322Ng_1^4g_2 + 60Ng_1^3g_2^2 - 31Ng_1^2g_2^3 - 125g_2^5) + \dots\end{aligned}\quad (2.1)$$

We have omitted the explicit three loop terms, which can be obtained from [23, 24]. Similarly, the anomalous dimensions of the fields  $\chi^i$  and  $\sigma$  take the form

$$\begin{aligned}\gamma_\chi &= \frac{g_1^2}{6(4\pi)^3} + \frac{g_1^2}{432(4\pi)^6}(g_1^2(11N+26) + 48g_1g_2 - 11g_2^2) + \dots, \\ \gamma_\sigma &= -\frac{Ng_1^2 - g_2^2}{12(4\pi)^3} - \frac{1}{432(4\pi)^6}(2Ng_1^4 + 48Ng_1^3g_2 - 11Ng_1^2g_2^2 - 13g_2^4) + \dots\end{aligned}\quad (2.2)$$

With the beta functions at hand, we can look for non-trivial fixed points of the RG flow satisfying  $\beta_1(g_1^*, g_2^*) = 0, \beta_2(g_1^*, g_2^*) = 0$ . For all positive  $N$ , we find two physically equivalent fixed points with purely imaginary coupling constants, hence all operator dimensions remain real. These fixed points are IR stable for all  $N$  (the stability matrix  $M_{ij} = \frac{\partial\beta_i}{\partial g_j}$  has positive eigenvalues). Note that this is different from the  $O(N)$  versions of these models [23, 24], where one finds a critical  $N$ , whose one loop value is  $\simeq 1038$ , below which the IR stable fixed points with real coupling constants disappear. However, to all orders in the  $1/N$  expansion, the fixed point couplings and conformal dimensions in the  $\text{Sp}(N)$  models are related to the ones in the  $O(N)$  models by the replacement  $N \rightarrow -N$ .

Figure 1 shows the RG flow directions for  $N = 2$ . The arrows indicate how the coupling constants flow towards the IR. The two IR fixed points are physically equivalent because they are related by  $g_i \rightarrow -g_i$ . At higher values of  $N$ , the qualitative behavior of the RG



**Figure 1.** The zeroes of the one loop  $\beta$  functions and the RG flow directions for the  $\text{OSp}(1|2)$  model. The coordinates are defined via  $g_1 = i\sqrt{\frac{(4\pi)^3\epsilon}{5}}x$ ,  $g_2 = i\sqrt{\frac{(4\pi)^3\epsilon}{5}}y$ , and the red dots correspond to the stable IR fixed points.

flows and fixed points remain the same. We still have a UV Gaussian fixed point, and two stable IR fixed points.

A special structure emerges for  $N = 2$ . In this case we find the fixed point solution

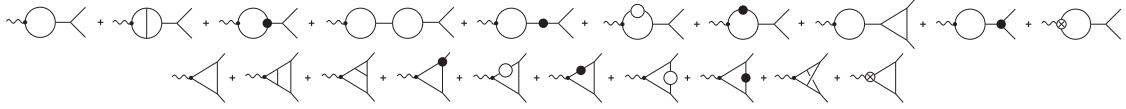
$$g_2^* = 2g_1^*, \quad g_1^* = i\sqrt{\frac{(4\pi)^3\epsilon}{5}} \left( 1 + \frac{67}{180}\epsilon + \mathcal{O}(\epsilon^2) \right), \quad (2.3)$$

and the conformal dimensions of the fundamental fields are equal (the three loop term given below can be obtained from the results in [23, 24])

$$\Delta_\sigma = \Delta_\chi = 2 - \frac{8}{15}\epsilon - \frac{7}{450}\epsilon^2 - \frac{269 - 702\zeta(3)}{33750}\epsilon^3 + \dots \quad (2.4)$$

We show in the next section that the equality of dimensions is a consequence of a symmetry enhancement from  $\text{Sp}(2)$  to the supergroup  $\text{OSp}(1|2)$ .

It is natural to ask whether symmetry enhancement can occur at other values of  $N$ . For instance, we can explicitly check for which  $N$  the dimensions of  $\sigma$  and  $\chi$  are equal. A direct calculation using the beta functions and anomalous dimensions up to three loops shows that this only happens for  $N = 2$  and  $N = -1$ . The latter case corresponds to the



**Figure 2.** Diagrams contributing to the mixing of  $\sigma^2$  and  $\Omega_{ij}\chi^i\chi^j$  operators to two loop order. Note that we do not distinguish the  $\sigma$  and  $\chi^i$  propagators, which are both denoted by solid lines. Therefore, each diagram represents several possible diagrams of the same topology. The solid dots represent one loop counter terms for either the propagator or the vertex, and the crossed dots represent one loop counter terms for the operator mixing. The wavy lines represent quadratic operator insertions.

3-state Potts model fixed point of the the theory with two commuting scalars [24]: it has  $g_1^* = -g_2^*$  and enhanced  $Z_3$  symmetry.<sup>7</sup>

The anomalous dimensions of some composite operators may be similarly obtained from the results in [23, 24]. Let us quote the explicit result for the quadratic operators arising from the mixture of  $\sigma^2$  and  $\Omega_{ij}\chi^i\chi^j$ . These operators have the same classical dimension, so we expect them to mix. The  $2 \times 2$  anomalous dimension mixing matrix  $\gamma^{ab}$  was given in [23, 24] up to one-loop order. Extending those results to two loops, we find the mixing matrix

$$\begin{pmatrix} -\frac{g_1^2(N+4)}{6(4\pi)^3} + \frac{g_1^2(g_1^2(21N-134)-6g_1g_2(2N+5)+5g_2^2)}{108(4\pi)^6} & \frac{Ng_1(6g_1-g_2)}{6(4\pi)^3} + \frac{Ng_1^2(18g_1^2+161g_1g_2+15g_2^2)}{108(4\pi)^6} \\ \frac{g_1(g_2-6g_1)}{6(4\pi)^3} - \frac{g_1(6g_1^3(11N+20)-g_1^2g_2(11N-324)+12g_1g_2^2-13g_2^3)}{216(4\pi)^6} & -\frac{Ng_1^2+4g_2^2}{6(4\pi)^3} + \frac{Ng_1^2(160g_1^2+36g_1g_2-41g_2^2)-181g_2^4}{216(4\pi)^6} \end{pmatrix} \quad (2.5)$$

where index ‘1’ corresponds to the operator  $\Omega_{ij}\chi^i\chi^j$ , and index ‘2’ corresponds to  $\sigma^2$ . The diagrams contributing to the calculation are listed in figure 2.

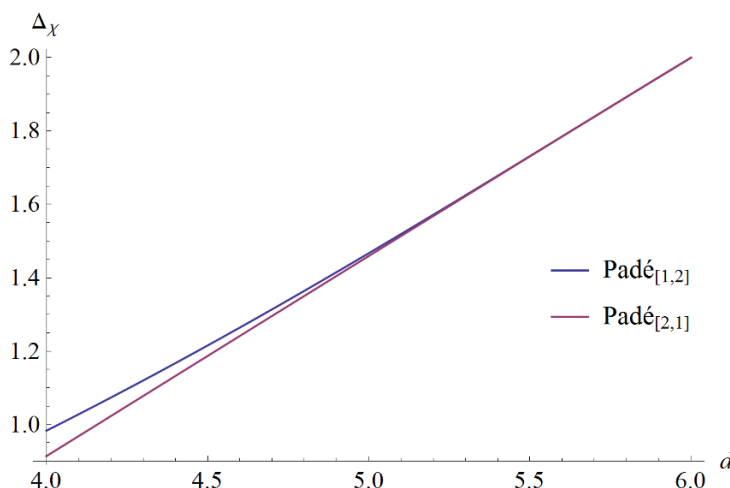
The two eigenvectors of  $\gamma^{ab}$  give the two linear combinations of  $\sigma^2$  and  $\Omega_{ij}\chi^i\chi^j$ , and the eigenvalues  $\gamma_{\pm}$  give their anomalous dimensions, so that  $\Delta_{\pm} = d - 2 + \gamma_{\pm}$ . We find that one of these combinations is a conformal primary, and the other one is a descendant of  $\sigma$ . Indeed, after plugging in the fixed point couplings, we find  $\Delta_- = \Delta_{\sigma} + 2$ . For instance, in the large  $N$  expansion we find the results

$$\begin{aligned} \Delta_- &= \Delta_{\sigma} + 2 = 4 + \frac{-40\epsilon + \frac{104}{3}\epsilon^2}{N} + \frac{6800\epsilon - \frac{34190}{3}\epsilon^2}{N^2} + \dots, \\ \Delta_+ &= 4 + \frac{100\epsilon - \frac{395}{3}\epsilon^2}{N} + \frac{-49760\epsilon + \frac{237476}{3}\epsilon^2}{N^2} + \dots \end{aligned} \quad (2.6)$$

Upon sending  $N \rightarrow -N$ , the dimension  $\Delta_+$  can be checked to be in agreement with the available large  $N$  results for the critical exponent  $\omega$  in the quartic  $O(N)$  theory<sup>8</sup> [32, 45].

<sup>7</sup>For  $N = -1$ , there is also a (non-unitary) solution with  $g_1^* = g_2^*$  which corresponds to two decoupled Fisher models [37], and hence the dimensions of the fundamental fields are trivially equal.

<sup>8</sup>This critical exponent is related to the derivative of the beta function in the  $O(N)$  theory with quartic interaction; see [24] for more details on the comparison to the large  $N$  results.



**Figure 3.** Different Padé approximations of  $\Delta_\chi$  for the  $N = 2$  model.

## 2.1 Estimating operator dimensions with Padé approximants

In the previous section we calculated  $\epsilon$  expansions for the operator dimensions of the cubic  $\text{Sp}(N)$  theory in  $d = 6 - \epsilon$ . We may use the following Padé approximant to estimate the behavior of operator dimensions as we continue  $d$ :

$$\text{Padé}_{m,n}[d] = \frac{A_0 + A_1 d + A_2 d^2 + \dots + A_m d^m}{1 + B_1 d + B_2 d^2 + \dots + B_n d^n}. \quad (2.7)$$

For  $N = 2$ , by demanding that the  $\epsilon$  expansion have the same behavior as (2.4), we can fix four coefficients in the Padé approximant. Here we use  $\text{Padé}_{1,2}$ , and  $\text{Padé}_{2,1}$ , to obtain the following estimate for the operator dimension in  $d = 5$ :

$$\Delta_\chi^{N=2} = \Delta_\sigma^{N=2} \approx \begin{cases} 1.467 & \text{with Padé}_{1,2} \\ 1.459 & \text{with Padé}_{2,1} \end{cases}. \quad (2.8)$$

As expected, this is below the unitarity bound in  $d = 5$ , which is  $\Delta = \frac{3}{2}$ . The plots of different Padé approximants are shown in figure 3. For the next primary operator, whose scaling dimension is given in (3.11), using the  $\text{Padé}_{2,1}$  approximant we estimate  $\Delta_+ \approx 3.35$  in  $d = 5$ . In four dimensions we estimate  $\Delta_+ \approx 2.7$ ; this suggests that the  $N = 2$  model does not have a free field description near  $d = 4$ .

In the  $\text{Sp}(4)$  case, we may assume for  $4 < d < 6$  that the IR fixed point of the cubic theory (1.3) is equivalent to the UV fixed point of the quartic theory (1.1).<sup>9</sup> If so, then not only do we know the  $6 - \epsilon$  expansion of  $\Delta_\chi$  to  $\mathcal{O}(\epsilon^3)$ , we can also use the quartic theory

<sup>9</sup>This equivalence holds for  $\text{Sp}(N)$  models with sufficiently large  $N$ , but it is not completely clear if it applies for  $N = 4$ .



result in  $d = 4 + \epsilon$ , which is known to  $\mathcal{O}(\epsilon^5)$ :

$$\Delta_\chi = \begin{cases} 2 - 0.529827\epsilon - 0.0126197\epsilon^2 + 0.0157244\epsilon^3 & \text{in } d = 6 - \epsilon \\ 1 + 0.5\epsilon - 0.03125\epsilon^2 + 0.015625\epsilon^3 - 0.0730951\epsilon^4 + 0.0195503\epsilon^5 & \text{in } d = 4 + \epsilon \end{cases} \quad (2.9)$$

Together, these expansions allow us to determine ten coefficients of the Padé approximant. Disregarding the approximants that have a pole, we obtain the estimate  $\Delta_\chi^{N=4} \approx 1.478$  in  $d = 5$ .

## 2.2 Sphere free energies

It is also interesting to compute the sphere free energy at the IR fixed point in  $d = 6 - \epsilon$ . The leading order term in the  $\epsilon$  expansion for the corresponding  $O(N)$  models was computed in [40]. Sending  $N \rightarrow -N$ , we find the following result for  $\tilde{F} = \sin(\frac{\pi d}{2}) \log Z_{S^d}$  in the cubic  $\text{Sp}(N)$  models

$$\tilde{F}_{\text{IR}} = \tilde{F}_{\text{UV}} - \frac{\pi}{17280} \frac{(g_2^*)^2 - 3N(g_1^*)^2}{(4\pi)^3} \epsilon + \mathcal{O}(\epsilon^3), \quad (2.10)$$

where  $\tilde{F}_{\text{UV}} = (1-N)\tilde{F}_s$ , and  $\tilde{F}_s$  the value corresponding to a free conformal scalar. Plugging in the explicit solutions for the fixed point couplings  $g_1^*, g_2^*$ , it is straightforward to verify that, for all  $N \geq 2$ , we have

$$\tilde{F}_{\text{UV}} > \tilde{F}_{\text{IR}}. \quad (2.11)$$

For instance, for  $N = 2$  we find

$$\tilde{F}_{\text{IR}} = -\tilde{F}_s - \frac{\pi}{43200} \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (2.12)$$

Note that for the Fisher model [37] of the Lee-Yang edge singularity, corresponding to  $N = 0$  and imaginary  $g_2^*$ , the inequality (2.11) does not hold. For general non-unitary theories the inequality does not have to hold; remarkably, it does hold for the  $\text{Sp}(N)$  models with positive  $N$ .

Similarly, using the results of [40] for the quartic  $O(N)$  theory in  $d = 4 - \epsilon$ , we can compute the sphere free energy at the IR fixed point of the model (1.1) in  $d = 4 - \epsilon$ . We find

$$\tilde{F}_{\text{quartic}} = \tilde{F}_{\text{free}} - \frac{\pi}{576} \frac{N(N-2)}{(N-8)^2} \epsilon^3 + \mathcal{O}(\epsilon^4), \quad (2.13)$$

where  $\tilde{F}_{\text{free}} = -N\tilde{F}_s$ . We observe that  $\tilde{F}_{\text{free}} > \tilde{F}_{\text{quartic}}$  is satisfied for all  $N > 2$ .<sup>10</sup>

The case  $N = 8$  is special and needs to be treated separately. Here the one-loop term in the beta function vanishes, and we have

$$\beta_\lambda = -\epsilon\lambda + \frac{15}{32\pi^4} \lambda^3 + \mathcal{O}(\lambda^4). \quad (2.14)$$

Therefore, at the IR fixed point  $\lambda_*^2 = \frac{32\pi^4}{15} \epsilon + \mathcal{O}(\epsilon^{3/2})$ , and we get

$$\tilde{F}_{\text{quartic}} = \tilde{F}_{\text{free}} - \frac{\pi}{360} \epsilon^2 + \dots \quad (2.15)$$

---

<sup>10</sup>For  $\epsilon < 0$  the interacting theory has a UV fixed point. So, in this case  $\tilde{F}_{\text{free}} < \tilde{F}_{\text{quartic}}$ , again in agreement with the conjectured  $\tilde{F}$  theorem.

### 3 Symmetry enhancement for $N = 2$

Let us write the cubic  $\text{Sp}(2)$  model in terms of a real scalar  $\sigma$  and a single complex anti-commuting fermion  $\theta$ :

$$S = \int d^d x \left( \partial_\mu \theta \partial^\mu \bar{\theta} + \frac{1}{2} (\partial_\mu \sigma)^2 + g_1 \sigma \theta \bar{\theta} + \frac{1}{6} g_2 \sigma^3 \right). \quad (3.1)$$

For  $g_2 = 2g_1$ , i.e. the fixed point relation (2.3), this action possesses a fermionic symmetry with a complex anti-commuting scalar parameter  $\alpha$

$$\delta \theta = \sigma \alpha, \quad \delta \bar{\theta} = \sigma \bar{\alpha}, \quad \delta \sigma = -\alpha \bar{\theta} + \bar{\alpha} \theta. \quad (3.2)$$

As a consequence of this symmetry, the scaling dimensions of  $\sigma$  and  $\theta$  are equal, as seen explicitly in eq. (2.4). This complex fermionic symmetry enhances the  $\text{Sp}(2)$  to  $\text{OSp}(1|2)$ , which is the smallest supergroup. The full set of supergroup generators can be given in the form

$$\begin{aligned} Q^+ &= \frac{1}{2} \left( \sigma \frac{\partial}{\partial \bar{\theta}} + \theta \frac{\partial}{\partial \sigma} \right), & Q^- &= \frac{1}{2} \left( \sigma \frac{\partial}{\partial \theta} - \bar{\theta} \frac{\partial}{\partial \sigma} \right), \\ J^+ &= \theta \frac{\partial}{\partial \bar{\theta}}, & J^- &= \bar{\theta} \frac{\partial}{\partial \theta}, & J^3 &= \frac{1}{2} \left( \theta \frac{\partial}{\partial \theta} - \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right), \end{aligned} \quad (3.3)$$

and it is not hard to check that they satisfy the algebra of  $\text{OSp}(1|2)$ :

$$[J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^3 \quad (3.4)$$

$$[J^3, Q^\pm] = \pm \frac{1}{2} Q^\pm, \quad [J^\pm, Q^\mp] = -Q^\pm \quad (3.5)$$

$$\{Q^\pm, Q^\pm\} = \pm \frac{1}{2} J^\pm, \quad \{Q^+, Q^-\} = \frac{1}{2} J^3 \quad (3.6)$$

We expect the operators of the theory to form representations of  $\text{OSp}(1|2)$ . For example, let us study the mixing of the  $\text{Sp}(2)$  singlet operators  $\sigma^2$  and  $\theta \bar{\theta}$ . Setting  $N = 2$  in (2.5), we find the scaling dimensions of the two eigenstates:

$$\Delta_+ = 4 - \frac{2}{3}\epsilon + \frac{1}{30}\epsilon^2 + \mathcal{O}(\epsilon^3), \quad \Delta_- = 4 - \frac{8}{15}\epsilon - \frac{7}{450}\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (3.7)$$

The first of these dimensions corresponds to the conformal primary operator

$$O_+ = \sigma^2 + 2\theta \bar{\theta}, \quad (3.8)$$

which is invariant under all the  $\text{OSp}(1|2)$  generators. The second corresponds to  $O_- = \sigma^2 + \theta \bar{\theta}$ , which is a conformal descendant because at the fixed point it is proportional to  $\partial_\mu \partial^\mu \sigma$  by equations of motion. Indeed, we find  $\Delta_- = \Delta_\sigma + 2$ .

Continuation of the results for the cubic model with  $\text{OSp}(1|2)$  global symmetry to finite  $\epsilon$  points to the existence of such interacting CFTs in integer dimensions below 6. In [38] it was argued that the  $q \rightarrow 0$  limit of the  $q$ -state Potts model is described by the

OSp(1|2) sigma model [46, 47]. There is evidence that the upper critical dimension of the spanning forest model is 6 [39], and the  $6 - \epsilon$  expansions of the critical indices are [39, 48]

$$\eta = -\frac{1}{15}\epsilon - \frac{7}{225}\epsilon^2 - \frac{269 - 702\zeta(3)}{16875}\epsilon^3 + \mathcal{O}(\epsilon^4), \quad (3.9)$$

$$\nu^{-1} = 2 - \frac{1}{3}\epsilon - \frac{1}{30}\epsilon^2 - \frac{173 - 864\zeta(3)}{27000}\epsilon^3 + \mathcal{O}(\epsilon^4). \quad (3.10)$$

Remarkably, the operator dimensions we have calculated (2.4) and (3.7) agree with these expansions upon the standard identifications

$$\Delta_\chi = \frac{d}{2} - 1 + \frac{\eta}{2}, \quad \Delta_+ = d - \nu^{-1}. \quad (3.11)$$

Similarly, we can match the  $6 - \epsilon$  expansions of the sphere free energies. For the  $q$ -state Potts model it is not hard to show that

$$\tilde{F}_q = (q-1)\tilde{F}_s + \frac{\pi(q-1)(q-2)}{8640(3q-10)}\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (3.12)$$

and for  $q = 0$  this matches the  $\tilde{F}$  of our OSp(1|2) model, (2.12). These results provide strong evidence that the OSp(1|2) symmetric IR fixed point of the cubic theory (3.1) describes the  $q \rightarrow 0$  limit of the  $q$ -state Potts model.

Numerical simulations of the spanning-forest model [39], which is equivalent to the  $q \rightarrow 0$  limit of the ferromagnetic  $q$ -state Potts model, indicate the existence of second-order phase transitions in dimensions 3, 4 and 5. The critical exponents found in [39] are in good agreement with the Pade extrapolations of  $6 - \epsilon$  expansions exhibited in section 2.1. This provides additional evidence for the existence of the critical theories with OSp(1|2) symmetry. It would be of further interest to find the critical statistical models that are described by the Sp( $N$ ) invariant theories with  $N > 2$ .

## Acknowledgments

We thank D. Anninos, D. Harlow and J. Maldacena, and especially S. Caracciolo and G. Parisi, for useful discussions. The work of LF and SG was supported in part by the US NSF under Grant No. PHY-1318681. The work of IRK and GT was supported in part by the US NSF under Grant No. PHY-1314198.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

## References

- [1] K.G. Wilson and M.E. Fisher, *Critical exponents in 3.99 dimensions*, *Phys. Rev. Lett.* **28** (1972) 240 [[INSPIRE](#)].
- [2] K.G. Wilson and J.B. Kogut, *The Renormalization group and the  $\epsilon$ -expansion*, *Phys. Rept.* **12** (1974) 75 [[INSPIRE](#)].

- [3] J.M. Maldacena, *The Large- $N$  limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](#)] [[INSPIRE](#)].
- [4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [6] I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical  $O(N)$  vector model*, *Phys. Lett. B* **550** (2002) 213 [[hep-th/0210114](#)] [[INSPIRE](#)].
- [7] E.S. Fradkin and M.A. Vasiliev, *On the gravitational interaction of massless higher spin fields*, *Phys. Lett. B* **189** (1987) 89 [[INSPIRE](#)].
- [8] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in  $(3+1)$ -dimensions*, *Phys. Lett. B* **243** (1990) 378 [[INSPIRE](#)].
- [9] M.A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in  $(3+1)$ -dimensions*, *Phys. Lett. B* **285** (1992) 225 [[INSPIRE](#)].
- [10] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in  $(A)dS(d)$* , *Phys. Lett. B* **567** (2003) 139 [[hep-th/0304049](#)] [[INSPIRE](#)].
- [11] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Annals Phys.* **144** (1982) 249 [[INSPIRE](#)].
- [12] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys. B* **556** (1999) 89 [[hep-th/9905104](#)] [[INSPIRE](#)].
- [13] S. Giombi and X. Yin, *The higher spin/vector model duality*, *J. Phys. A* **46** (2013) 214003 [[arXiv:1208.4036](#)] [[INSPIRE](#)].
- [14] A. Strominger, *The dS/CFT correspondence*, *JHEP* **10** (2001) 034 [[hep-th/0106113](#)] [[INSPIRE](#)].
- [15] D. Anninos, T. Hartman and A. Strominger, *Higher spin realization of the dS/CFT correspondence*, [arXiv:1108.5735](#) [[INSPIRE](#)].
- [16] A. LeClair, *Quantum critical spin liquids, the 3D Ising model and conformal field theory in  $2+1$  dimensions*, [cond-mat/0610639](#) [[INSPIRE](#)].
- [17] A. LeClair and M. Neubert, *Semi-Lorentz invariance, unitarity and critical exponents of symplectic fermion models*, *JHEP* **10** (2007) 027 [[arXiv:0705.4657](#)] [[INSPIRE](#)].
- [18] A. Higuchi, *Forbidden Mass Range for Spin-2 Field Theory in de Sitter Space-time*, *Nucl. Phys. B* **282** (1987) 397 [[INSPIRE](#)].
- [19] A. Higuchi, *Symmetric Tensor Spherical Harmonics on the  $N$  Sphere and Their Application to the de Sitter Group  $SO(N,1)$* , *J. Math. Phys.* **28** (1987) 1553 [Erratum *ibid.* **43** (2002) 6385] [[INSPIRE](#)].
- [20] S. Deser and A. Waldron, *Partial masslessness of higher spins in  $(A)dS$* , *Nucl. Phys. B* **607** (2001) 577 [[hep-th/0103198](#)] [[INSPIRE](#)].
- [21] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013 [[astro-ph/0210603](#)] [[INSPIRE](#)].

- [22] D. Anninos, F. Denef and D. Harlow, *Wave function of Vasiliev's universe: a few slices thereof*, *Phys. Rev. D* **88** (2013) 084049 [[arXiv:1207.5517](#)] [[INSPIRE](#)].
- [23] L. Fei, S. Giombi and I.R. Klebanov, *Critical  $O(N)$  models in  $6 - \epsilon$  dimensions*, *Phys. Rev. D* **90** (2014) 025018 [[arXiv:1404.1094](#)] [[INSPIRE](#)].
- [24] L. Fei, S. Giombi, I.R. Klebanov and G. Tarnopolsky, *Three loop analysis of the critical  $O(N)$  models in  $6 - \epsilon$  dimensions*, *Phys. Rev. D* **91** (2015) 045011 [[arXiv:1411.1099](#)] [[INSPIRE](#)].
- [25] G. Parisi, *The Theory of Nonrenormalizable Interactions. 1. The Large- $N$  Expansion*, *Nucl. Phys. B* **100** (1975) 368 [[INSPIRE](#)].
- [26] G. Parisi, *On non-renormalizable interactions*, in *New Developments in Quantum Field Theory and Statistical Mechanics Cargèse 1976*, Springer, New York U.S.A. (1977), pg. 281.
- [27] X. Bekaert, E. Meunier and S. Moroz, *Towards a gravity dual of the unitary Fermi gas*, *Phys. Rev. D* **85** (2012) 106001 [[arXiv:1111.1082](#)] [[INSPIRE](#)].
- [28] E. Ma, *Asymptotic Freedom and a Quark Model in Six-Dimensions*, *Prog. Theor. Phys.* **54** (1975) 1828 [[INSPIRE](#)].
- [29] A.N. Vasiliev, M. Pismak, Yu and Yu. R. Khonkonen, *Simple Method of Calculating the Critical Indices in the  $1/N$  Expansion*, *Theor. Math. Phys.* **46** (1981) 104 [[INSPIRE](#)].
- [30] A.N. Vasiliev, Yu. M. Pismak and Yu. R. Khonkonen,  *$1/N$  expansion: calculation of the exponent  $\eta$  in the order  $1/N^3$  by the conformal bootstrap method*, *Theor. Math. Phys.* **50** (1982) 127 [[INSPIRE](#)].
- [31] K. Lang and W. Rühl, *Field algebra for critical  $O(N)$  vector nonlinear  $\sigma$ -models at  $2 < d < 4$* , *Z. Phys. C* **50** (1991) 285 [[INSPIRE](#)].
- [32] K. Lang and W. Rühl, *The critical  $O(N)$   $\sigma$ -model at dimensions  $2 < d < 4$ : Fusion coefficients and anomalous dimensions*, *Nucl. Phys. B* **400** (1993) 597 [[INSPIRE](#)].
- [33] A. Petkou, *Conserved currents, consistency relations and operator product expansions in the conformally invariant  $O(N)$  vector model*, *Annals Phys.* **249** (1996) 180 [[hep-th/9410093](#)] [[INSPIRE](#)].
- [34] Y. Nakayama and T. Ohtsuki, *Five dimensional  $O(N)$ -symmetric CFTs from conformal bootstrap*, *Phys. Lett. B* **734** (2014) 193 [[arXiv:1404.5201](#)] [[INSPIRE](#)].
- [35] S.M. Chester, S.S. Pufu and R. Yacoby, *Bootstrapping  $O(N)$  vector models in  $4 < d < 6$* , *Phys. Rev. D* **91** (2015) 086014 [[arXiv:1412.7746](#)] [[INSPIRE](#)].
- [36] J.-B. Bae and S.-J. Rey, *Conformal Bootstrap Approach to  $O(N)$  Fixed Points in Five Dimensions*, [arXiv:1412.6549](#) [[INSPIRE](#)].
- [37] M.E. Fisher, *Yang-Lee Edge Singularity and  $\phi^3$  Field Theory*, *Phys. Rev. Lett.* **40** (1978) 1610 [[INSPIRE](#)].
- [38] S. Caracciolo, J.L. Jacobsen, H. Saleur, A.D. Sokal and A. Sportiello, *Fermionic field theory for trees and forests*, *Phys. Rev. Lett.* **93** (2004) 080601 [[cond-mat/0403271](#)] [[INSPIRE](#)].
- [39] Y. Deng, T.M. Garoni and D. Sokal, *Ferromagnetic phase transition for the spanning-forest model ( $q \rightarrow 0$  limit of the Potts model) in three or more dimensions*, *Phys. Rev. Lett.* **98** (2007) 030602 [[cond-mat/0610193](#)] [[INSPIRE](#)].
- [40] S. Giombi and I.R. Klebanov, *Interpolating between  $a$  and  $F$* , *JHEP* **03** (2015) 117 [[arXiv:1409.1937](#)] [[INSPIRE](#)].

- [41] H. Casini, M. Huerta and R.C. Myers, *Towards a derivation of holographic entanglement entropy*, *JHEP* **05** (2011) 036 [[arXiv:1102.0440](#)] [[INSPIRE](#)].
- [42] D.L. Jafferis, I.R. Klebanov, S.S. Pufu and B.R. Safdi, *Towards the F-theorem:  $\mathcal{N} = 2$  field theories on the three-sphere*, *JHEP* **06** (2011) 102 [[arXiv:1103.1181](#)] [[INSPIRE](#)].
- [43] I.R. Klebanov, S.S. Pufu and B.R. Safdi, *F-theorem without supersymmetry*, *JHEP* **10** (2011) 038 [[arXiv:1105.4598](#)] [[INSPIRE](#)].
- [44] S. Giombi, I.R. Klebanov and B.R. Safdi, *Higher Spin  $AdS_{d+1}/CFT_d$  at One Loop*, *Phys. Rev. D* **89** (2014) 084004 [[arXiv:1401.0825](#)] [[INSPIRE](#)].
- [45] D.J. Broadhurst, J.A. Gracey and D. Kreimer, *Beyond the triangle and uniqueness relations: Nonzeta counterterms at large- $N$  from positive knots*, *Z. Phys. C* **75** (1997) 559 [[hep-th/9607174](#)] [[INSPIRE](#)].
- [46] N. Read and H. Saleur, *Exact spectra of conformal supersymmetric nonlinear  $\sigma$ -models in two-dimensions*, *Nucl. Phys. B* **613** (2001) 409 [[hep-th/0106124](#)] [[INSPIRE](#)].
- [47] H. Saleur and B. Wehefritz Kaufmann, *Integrable quantum field theories with supergroup symmetries: The  $OSP(1/2)$  case*, *Nucl. Phys. B* **663** (2003) 443 [[hep-th/0302144](#)] [[INSPIRE](#)].
- [48] O.F. de Alcantara Bonfim, J.E. Kirkham and A.J. McKane, *Critical Exponents to Order  $\epsilon^3$  for  $\phi^3$  Models of Critical Phenomena in  $6 - \epsilon$ -dimensions*, *J. Phys. A* **13** (1980) L247 [Erratum *ibid.* **A 13** (1980) 3785] [[INSPIRE](#)].